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Parametric Representation of Anatomical Structures of the Human Body by Means of Trigonometric Interpolating Sums

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An approach for the parametric representation of anatomical structures of the human body by means of trigonometric interpolating sums (TIS) is introduced. This representation is constructed on the basis of the geometric information provided by medical digital images and an arbitrarily chosen system of curvilinear coordinates. The parameterization defined by these coordinates is approximated through TIS by using a multidimensional extension of the Lanczos's method for accelerating the convergence of trigonometric approximations for smooth, nonperiodic functions. This allows us to obtain accurate representations for a wide class of anatomical structures, including nonclosed ones. An upper bound of the approximation error is derived in the sense of the supremum norm. The reconstruction of a human face and the surface of a brain cortex are shown as illustrative examples of the parameterization by means of TIS.

1. INTRODUCTION

Different imaging techniques, such as computed tomography (CT), magnetic resonance imaging (MRI) and ultrasound imaging, give us morphological information about the human body and its internal organs (e.g., contours, surfaces, 3D objects). The mathematical representation of this geometric information is a key issue in various areas of medical physics. Examples are the computation of dose distribution in radiotherapy, the volume of organs and lesions [16], the forward-and-inverse problem in electrocardiography [22] and electroencephalography [6], etc.

It is not possible generally to obtain exact and explicit mathematical representations of anatomical structures. However, the digital morphological data drawn from these imaging techniques (i.e., the coordinates of large discrete sets of points on anatomical structures) provides abundant empirical information for the construction of approximate representations.

The most simple and widely used methods for image representation are those based on voxels or triangulation [5, 9, 19]. Although these methods show good performance in many biomedical image processing and analysis tasks, their utility is hampered by large storage requirements, the absence of the concepts of contour, surface, and object parameterizations and the fact that they are not useful in applications where higher order continuity of the representation is required.

An alternative approach for the representation of geometric objects is the use of trigonometric interpolating sums (TIS) [1, 3, 13, 15, 17, 23], which has well-known advantages. It gives parametric and explicit representations (i.e., analytic parameterizations) which can be used as a basis, not only for picture processing, but also for mathematical analysis. There are efficient algorithms for their computation, namely, the fast Fourier transforms. Also, they allow for a simple control of the trade-off between representation accuracy and computational cost through the selection of the number and location of the sample points and the set of terms retained in the trigonometric sums.

However, the use of TIS for the representation of geometric objects has been limited due to the fact that these objects are frequently nonclosed. Thus, its parameterizations are nonperiodic functions. This tends to produce poor approximations for moderate computational costs because the convergence rate of the TIS is, in general, very slow for nonperiodic functions [8]. Some attempts have been made to overcome this difficulty if they are restricted to surfaces that are nonclosed in only one of its directions [13, 17].

In this paper a general approach that solves this limitation is introduced. This leads to TIS approximations with good convergence properties even for nonclosed anatomical structures in multiple directions. The basis of this approach is the extension to multidimensional functions of the Lanczos's method [8, 10, 11] for accelerating the convergence of TIS for smooth, nonperiodic functions of one variable.

In Section 2 Lanczos' method is described and an extension to functions of several variables is introduced. An upper bound for the approximation error resulting from the use of the extended Lanczos' method is derived. Section 3 deals with the application of this method for the representation of anatomical structures. In the last section, the reconstruction of a human face and a piece of the brain cortex surface are shown as illustrative examples of representations by means of TIS.

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2. TRIGONOMETRIC INTERPOLATING SUMS FOR THE APPROXIMATION OF SMOOTH REAL-VALUED FUNCTIONS

This section discusses methods for accelerating the convergence of the approximations by means of TIS for smooth functions for which there are no smooth periodic extensions.

Consider a smooth real-world function ϕ of p variables defined on a p-dimensional cube $V = [-1, 1] \times \cdots \times [-1, 1]$. For $i = 1, ..., p$, let N_i be given positive integer numbers. The sum

$$\Phi(v_1, ..., v_p) = \sum_{k_1,...,k_p} C(\phi)_{k_1,...,k_p} \times \exp\left(\sum_{i=1}^{p} j\pi k_i(v_i + 1)\right), \tag{2.1}$$

with

$$C(\phi)_{k_1,...,k_p} = 1/T \sum_{t_1,...,t_p} \phi(s_1 - 1, ..., s_p - 1) \times \exp\left(-\sum_{i=1}^{p} j\pi k_i s_i\right), \tag{2.2}$$

defines the TIS relative to the set $Q(N_1, ..., N_p) = \{(v_1, ..., v_p) \in V\colon v_i = t_i/N_i - 1,\ t_i = 0, ...\ 2N_i - 1,\ i = 1\ ...\ p\}$ of $T = \Pi_{i=1}^{p} (2N_i)$ points in V. In expressions (2.1) and (2.2), k_i, $t_i = 0\ ...\ 2N_i - 1$, $s_i = t_i/N_i$, $i = 1\ ...\ p$, and $j = \sqrt{-1}$.

The expression (2.2) is the well-known p-dimensional discrete Fourier transform of the vector with components $\phi(q)$, q belonging to $Q(N_1, ..., N_p)$. Its inverse transform is given by (2.1). These transforms are widely used due to the existence of efficient algorithms for their computation [2, 4, 12, 21].

The TIS Φ defined by (2.1) is an interpolating function for ϕ at the set of nodes $Q(N_1, ..., N_p)$, i.e., $\Phi(q) = \phi(q)$ at each point q of $Q(N_1, ..., N_p)$. It is well-known that if ϕ admits a smooth periodic extension, then Φ rapidly converges to ϕ as N_i tends to infinity for all $i = 1, ..., p$. However, if the boundary behavior of ϕ is such that its definition cannot be smoothly extended, then the convergence of its TIS could be very slow, so Φ could give a poor approximation. In this case Φ could show what is known as Gibb's phenomenon [14], i.e., high frequency oscillations between the interpolating nodes.

In the one-dimensional case, Lanczos has introduced a method to attenuate this undesirable effect [8, 10, 11], which is recalled in Part 1 of this section. In Part 2, this method is extended for functions of several variables.

2.1. *Lanczos's Method for the Approximation of Smooth Real-Valued Functions of One Variable by Means of TIS*

Let ψ be a real-valued function defined on $U = [0, 1]$ with a bounded derivative of order $2m + 1$ on U. Consider the problem of the approximation of ψ on the basis of the knowledge of the values $\psi(i/N)$, $\psi^k(0)$, and $\psi^k(1)$, for each $i = 0, ..., N$ and $k = 2, 4, ..., 2m$. Here ψ^k denotes the kth derivative of ψ.

The basic idea of the Lanczos's method is to construct an approximation to ψ by means of the TIS for an auxiliary periodic function ϕ, which has on its whole definition domain (including the boundary) the same smoothness degree of ψ. That is,

(C1) ϕ has a bounded derivative of order $2m + 1$

(C2) $\phi^k(-1) = \phi^k(1)$ for $k = 0, ..., 2m + 1$.

It is well-known that under these conditions the TIS Φ for ϕ,

$$\Phi(u) = \sum_{k=0}^{2N-1} C_k \exp(j\pi k(u + 1)), \tag{2.3}$$

with

$$C_k = 1/(2N) \sum_{t=0}^{2N-1} \phi(t/N - 1) \exp(-j\pi kt/N), \tag{2.4}$$

rapidly converges to ϕ [7, 8]. Indeed, there exists a constant C such that

$$|\phi(v) - \Phi(v)| \leq C \log(N)/N^{2m+1} \tag{2.5}$$

for all v on V.

Therefore, if the auxiliary function is defined by

$$\phi(v) = \begin{cases} \psi(v) - h(v) & \text{for } 0 \leq v \leq 1 \\ -\phi(-v) & \text{for } -1 \leq v \leq 0, \end{cases} \tag{2.6}$$

where h is a function such that the conditions (C1) and (C2) hold, and if the approximating function Ψ to ψ is defined by

$$\Psi(u) = \Phi(u) + h(u), \tag{2.7}$$

then it holds that

$$|\psi(u) - \Psi(u)| \leq C \log(N)/N^{2m+1} \tag{2.8}$$

for all u of U. Thus, the resulting approximation Ψ rapidly converges to ψ, therefore overcoming the slow convergence of TIS for nonperiodic functions.

Lanczos's method is defined by (2.6) and (2.7), setting the function h as the polynomial

$$h(u) = \sum_{k=0}^{m} [\psi^{2k}(0)P_{2k+1}(1-u) + \psi^{2k}(1)P_{2k+1}(u)]. \quad (2.9)$$

Here P_i are polynomials on [0, 1] such that $P_0(u) = 1$, $P_1(u) = u$, $dP_k(u)/du = P_{k-1}(u)$ for $k = 2, 3, \ldots$, and $P_{2k+1}(0) = P_{2k+1}(1) = 0$ for all $k = 1, 2, \ldots$. (Notice that $h^{2k}(0) = \psi^{2k}(0)$ and $h^{2k}(1) = \psi^{2k}(1)$ for $k = 0, \ldots, m$.)

It is worth noting that the symmetry of the auxiliary function ϕ reduces the cost of the numerical computation of Φ. Indeed, since ϕ is an odd real-valued function on V, the sums (2.3) and (2.4) are pure sine sums,

$$\sum_{k=1}^{N-1} b_k \sin(k\pi(v+1))$$

with

$$b_k = (2/N) \sum_{t=1}^{N-1} \phi(t/N - 1) \sin(k\pi t/N).$$

Thus, the coefficient b_k can be obtained by means of efficient algorithms for computing the sine transform [20].

2.2. *Extension of Lanczos' Method to Smooth Real-Valued Functions of Several Variables*

Let ψ be a real-valued function on the p-dimensional cube $U = [0,1] \times \cdots \times [0,1]$ with bounded partial derivatives $\psi^{k_1+\cdots+k_p} = \partial^{k_1+\cdots+k_p}\psi(u_1, \ldots, u_p)/\partial u_1^{k_1} \cdots \partial u_p^{k_p}$ for all $k_i = 0, \ldots, 2m+1$ and $i = 1, \cdots, p$. Given positive integer numbers $N_1, \ldots, N_p$, consider the problem of approximating ψ on the basis of its values over the grid of points $K(N_1, \ldots, N_p) = \{(u_1, \ldots, u_p) \in U: u_i = t_i/N_i, t_i = 0, \ldots, N_i, i = 1, \cdots, p\}$, and the values of the derivatives $\psi^{k_1+\cdots+k_p}$ ($k_i = 0, 2, \ldots, 2m$) at the boundary of U.

The extension of Lanczos's method to this multidimensional setting will be based on the construction of an auxiliary function ϕ as the odd extension to the p-dimensional cube $V = [-1,1] \times \cdots \times [-1,1]$ of $\psi - h_{p,m}$, where $h_{p,m}$ is a polynomial on U. This polynomial is defined in such a way that the following conditions are satisfied (compare with (C1) and (C2)):

(C3) the partial derivatives of order $2m+1$ of ϕ with respect to each argument exist and are bounded on V,

(C4) $\phi^{k_i}(v_1, \ldots, v_{i-1}, -1, v_{i+1}, \ldots, v_p) = \phi^{k_i}(v_1, \ldots, v_{i-1}, 1, v_{i+1}, \ldots, v_p)$ for $k_i = 0, \ldots, 2m+1$, and $i = 1, \ldots, p$.

Specifically, $h_{p,m}$ is defined by the relations

$$h_{i,m}(u) = (W_{i,m}[\psi - h_{i-1,m}])(u) + h_{i-1,m}(u) \quad (2.10)$$

for $i = 2, \ldots, p$, and $h_{1,m}(u) = (W_{1,m}\psi)(u)$. Here, $W_{i,m}$ is the operator on the space of differentiate functions on U defined by

$$\begin{aligned}(W_{i,m}f)(u_1, \ldots, u_p) &= \sum_{k_i=0}^{m} [f^{2k_i}(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_p)P_{2k_i+1}(1-u_i)) \\ &\quad + f^{2k_i}(u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_p)\, P_{2k_i+1}(u_i)], \quad (2.11)\end{aligned}$$

where $P_k(u)$ is the polynomial on [0, 1] of order k defined in Section 2.1. The verification of the conditions (C3) and (C4) for the auxiliary function just introduced can be seen in Appendix I.

The approximation Ψ to ψ by means of TIS is then defined by

$$\Psi(u) = \Phi(u) + h_{p,m}(u) \quad \text{for all } u \text{ on } U, \quad (2.12)$$

where Φ is the TIS of the auxiliary function ϕ. It can be shown (see Appendix II) that the error of this approximation is bounded according to

$$|\psi(u) - \Psi(u)| \leq \max_{i=1\cdots p} \{C_i \log(N_i)/N_i^{2m+1}\}, \quad (2.13)$$

for all u of U, where the constants C_i do not depend on u (compare with (2.8)).

The approximation just introduced requires the knowledge of the values of the functions $\psi^{k_1+\cdots+k_p}$ at each point of the boundary of U, for all $k_i = 0, 2, \ldots, 2m$, $i = 1, \ldots, p$. These functions, however, are frequently known only over the points of the boundary of U that belong to the finite set of nodes $K(N_1, \ldots, N_p)$. This is just the situation that has to be faced in order to reconstruct anatomical objects on the basis of discrete data obtained through medical imaging techniques (as will be discussed in Section 3).

In this case, the approximation Ψ to ψ is defined in the following recurrent way:

$$\Psi(u) = \overline{\Phi}(u) + H_{p,m_p}(u) \quad \text{for all } u \text{ on } U, \quad (2.14)$$

where $m_p \leq m$, $\overline{\Phi}$ is the TIS of the odd prolongation on V of the function $\psi - H_{p,m_p}$, and H_{p,m_p} is the approximation to the function h_{p,m_p} (defined by (2.10)–(2.11)) performed in the following way: the same type of approximation (2.14) is applied to each of the n-variables ($n = 1, \ldots, p-1$) functions that multiply the polynomials in the expression (2.11) involved in the definition of h_{p,m_p}. This recurrent algorithm finally reduced the computation of Ψ to the approximation of functions of one variable, for which the (nonrecurrent) approximation (2.7) is applied.

It can be shown that the resulting approximation error satisfies

$$|\psi(u) - \Psi(u)| \leq \max_{i,n=1\cdots p} \{C_{i,n} \log(N_i)/N_i^{2m_n+1}\}, \qquad (2.15)$$

for all u of U, where $C_{i,n}$ are some constants (see Appendix III). Here m_n denotes the order of the derivative used for the approximations of the n-dimensional functions involved in the approximation of the function h_{p,m_p} ($n = 1, ..., p-1; \sum_{n=1}^{p} m_n = m$) as a consequence of the consecutive steps of the recurrent approximation procedure just described.

The approximation (2.14) can be used for the accurate representation of anatomical structures, as discussed in the next section.

3. INTERPOLATING TRIGONOMETRIC SUMS FOR THE REPRESENTATION OF ANATOMICAL STRUCTURES

The morphological information provided by medical imaging techniques about an anatomical structure M (i.e., a contour, surface, or 3D object of the human body) consists of the coordinates (x, y, z) of a finite set G of points in M. It is desired to compute a parametric representation of M on the basis of these data.

For this purpose, choose an arbitrary smooth parameterization of M, given by a smooth map $(x(u), y(u), z(u))$ from the p-dimensional cube $U = [0, 1] \times \cdots \times [0, 1]$ onto M, such that it transforms the set of node $K(N_1, ..., N_p)$ on U onto G. Here $p = 1$ for a contour, $p = 2$ for a surface, and $p = 3$ for a 3D object.

Usually the parameterization regarded is not explicitly known. However, it is frequently easy to specify the correspondence that it induces between the finite sets of nodes $K(N_1, ..., N_p)$ and G, i.e., to give the values $(x(k), y(k), z(k))$ of the parameterization at the discrete nodes k of $K(N_1, ..., N_p)$.

This sets the problem of the construction of an approximation $(X(u), Y(u), Z(u))$ of the parameterization $(x(u), y(u), z(u))$ using its known values at the discrete nodes, i.e., such that the constraint

$$(X(k), Y(k), Z(k)) = (x(k), y(k), z(k)) \qquad (3.1)$$

is satisfied for all k in $K(N_1, ..., N_p)$.

Under suitable smoothness conditions (which will be assumed henceforth), the approximation of each coordinate function $X(u)$, $Y(u)$, and $Z(u)$ can be approached by applying the method of approximation through TIS that was developed in Section 2. The bound given there for the approximation error ensures that the approximate parameterization $(X(u), Y(u), Z(u))$ tends to the exact one as the number of nodes increases.

To disregard the information on the derivatives of the parameterization means to set $m = 0$ in the formulae of Section 2. Notice that in this case (3.1) expressed all the information about the parameterization that is required to construct its approximation. This case is developed in some detail below for smooth contours and surfaces.

3.1. *Representation of Arbitrary Smooth Contours*

Given the coordinates of $N + 1$ observed points of a contour M, the selection of the parameterization consists of the specification of an orientation and a speed of movement along the contour in such a way that (3.1) holds. The coordinate functions $x(u)$, $y(u)$, and $z(u)$ of the chosen parameterization are real-valued functions of one variable (i.e., $p = 1$) on $U = [0, 1]$.

In this case, the application of the methodology in Section 2 to approximate the coordinate functions $x(u)$, $y(u)$, and $z(u)$ reduces itself to the Lanczos's method. Thus, the approximations are computed according to expression (2.7) with $m = 0$.

3.2. *Representation of Arbitrary Smooth Surfaces*

Given a set G of $(N_1 + 1)(N_2 + 1)$ sample points of a (smooth) surface M of an anatomical structure, the selection of the parametric representation involves the specification of coordinate functions $x(u_1, u_2)$, $y(u_1, u_2)$, and $z(u_1, u_2)$ defined on the square $U = [0, 1] \times [0, 1]$ in such a way that $K(N_1, N_2)$ is transformed onto G.

Since now the coordinate functions are multidimensional, their approximation by means of the methodology introduced in Section 2 implies the use of the extended Lanczos's method (Section 2.2). In what follows this method is explicitly particularized for these two-variable functions setting ($p = 2$), with $m = 0$ ($m_1 = m_2 = m$). For simplicity, we write s and t instead of u_1 and u_2, respectively. Also, $\psi(s, t)$ will denote any one of the coordinate functions and $\Psi(s, t)$ will denote its approximating function.

The function $\Psi(s, t)$ is given by

$$\Phi(s, t) + H(s, t), \qquad (3.2)$$

where

$$\Phi(s, t) = \sum_{k=0}^{2N_1-1} \sum_{i=0}^{2N_2-1} C_{k,i} \exp(j\pi k(s+1)) \times \exp(j\pi i(t+1)), \qquad (3.3)$$

$$C_{k,i} = (1/(2N_2)) \sum_{n=0}^{2N_2-1} C_k(n/N_2 - 1) \times \exp(-j\pi in/N_2), \qquad (3.4)$$

and

$$C_k(t) = (1/(2N_1)) \sum_{n=0}^{2N_1-1} \phi(n/N_1 - 1, t) \times \exp(-j\pi kn/N_1), \qquad (3.5)$$

Here $\phi(v_1, v_2)$ is the auxiliary function defined on $V = [-1, 1] \times [-1, 1]$ by

$$\phi(v_1, v_2) = \begin{cases} \psi(v_1, v_2) - h_{p,m}(v_1, v_2) & \text{for } 0 \le v_1 \le 1, 0 \le v_2 \le 1 \\ -\phi(-v_1, v_2) & \text{for } -1 \le v_1 \le 0, 0 \le v_2 \le 1 \\ -\phi(v_1, -v_2) & \text{for } 0 \le v_1 \le 1, -1 \le v_2 \le 0 \\ \phi(-v_1, -v_2) & \text{for } -1 \le v_1 \le 0, -1 \le v_2 \le 0, \end{cases} \tag{3.6}$$

and $H_{p,m}(s, t)$ is an approximation to the function (see (2.10) and (2.11))

$$h_{p,m}(s, t) = (W_{2,0}\kappa)(s, t) + (W_{1,0}\psi)(s, t),$$

with $\kappa(s, t) = \psi(s, t) - (W_{1,0}\psi)(s, t)$, carried out as discussed in Section 2.2. The computation of this approximation is greatly simplified in this particular case because it is reduced to the approximation of one-variable functions, and thus the classical Lanczos's method is applied.

Applying (3.2) to each coordinate function, the approximation $(X(s, t), Y(s, t), Z(s, t))$ to the parameterization $(x(s, t), y(s, t), z(s, t))$ is obtained.

4. APPLICATIONS

In order to illustrate the actual performance of the methods introduced, they will be applied to construct approximate representations of a human face and the surface of a brain cortex on the basis of the morphological cross-sectional information provided by tomographic images.

For this purpose, a kind of parameterization that is applicable to a wide class of surfaces M in the 3D space will be introduced. Assume that the direction z of the 3D space is chosen in such a way that the intersection of each plane perpendicular to z with M is a smooth planar curve C_z. Consider that each curve C_z can be counterclockwise parameterized. Then a system of curvilinear coordinates on M can be defined by the following continuous map: to each point (s, t) on $U = [0, 1] \times [0, 1]$ there corresponds the point $(x(s, t), y(s, t), z(s))$ on M, where s determines the value of the coordinate z of the points on C_z (i.e., the movement along the z axis is parameterized by s), and for a fixed s, the values of the x and y coordinates of the points along C_z are counterclockwise parameterized by t (i.e., $x = x(s, t)$ and $y = y(s, t)$, where $z(s) = z$ and t describes the movement along C_z). The velocities of the movements along the s and t directions can be arbitrarily chosen.

The geometric information about M obtained from a tomograph is given by the coordinates (x, y, z) of $N_1 + 1$ points on $N_2 + 1$ serial cross-sectional contours which are plane-parallel with each other. This means the coordinates of $N_1 + 1$ points on each of $N_2 + 1$ curves C_z of M.

Following the methodology introduced in Section 3, these points can always be regarded as the image by the chosen parameterization $(x(s, t), y(s, t), z(s))$ of the nodes $K(N_1, N_2)$ on the two-dimensional unit cube U. That is, the desired approximation $(X(s, t), Y(s, t), Z(s))$ of the parameterization has to be determined on the basis of the interpolation condition (3.1):

$$(X(s_i, t_i), Y(s_i, t_i), Z(s_i)) = (x(s_i, t_i), y(s_i, t_i), z(s_i))$$

for all (s_i, t_i) in $K(N_1, N_2)$, where $(x(s_i, t_i), y(s_i, t_i), z(s_i))$ are the coordinates of the data points.

For each coordinate function, this interpolation problem is solved by means of TIS. Specifically, the approximation $Z(s)$ to $z(s)$ is computed according to (2.7), and the approximations $X(s, t)$ and $Y(s, t)$ to $x(s, t)$ and $y(s, t)$ are computed, respectively, according to (3.2).

Figures 1 and 2 illustrate the application of the representation by means of TIS to the reconstruction of anatomical structures from MRI data. Figure 1 shows the reconstruction of a piece of the surface of a human brain cortex. In this case the representation was approximated on the basis of 256 sample points on each of 30 serial cross sections. Figure 2 shows the reconstruction of a human face from the coordinates of 85 sample points on each of 85 cross sections.

These examples demonstrate the flexibility of the methodology introduced for the accurate representation of complex real objects with low computational cost. Particularly, the use of multidimensional extension of the Lanczos's method allows for accurate representations of even surfaces that are nonclosed in both dimensions, as is illustrated by the open surface (human face) presented in Fig. 2.

APPENDIX I: PROPERTIES OF THE AUXILIARY FUNCTION ϕ

In order to derive the properties (C3) and (C4) stated in Section 2.2 for the auxiliary function ϕ, we first prove that

$$h_{p,m}^{k_i}(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_p) = \psi^{k_i}(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_p) \tag{I.1}$$

and

$$h_{p,m}^{k_i}(u_1, ..., u_{i-1}, 1, u_{i+1}, ..., u_p) = \psi^{k_i}(u_1, ..., u_{i-1}, 1, u_{i+1}, ..., u_p) \tag{I.2}$$

for $k_i = 0, 2, ..., 2m$, $i = 1 ... p$, and u on U.

From the definition of $h_{p,m}$ and the assumptions made on ψ, the k_ith partial derivative $h_{p,m}^{k_i}$ with respect to variable

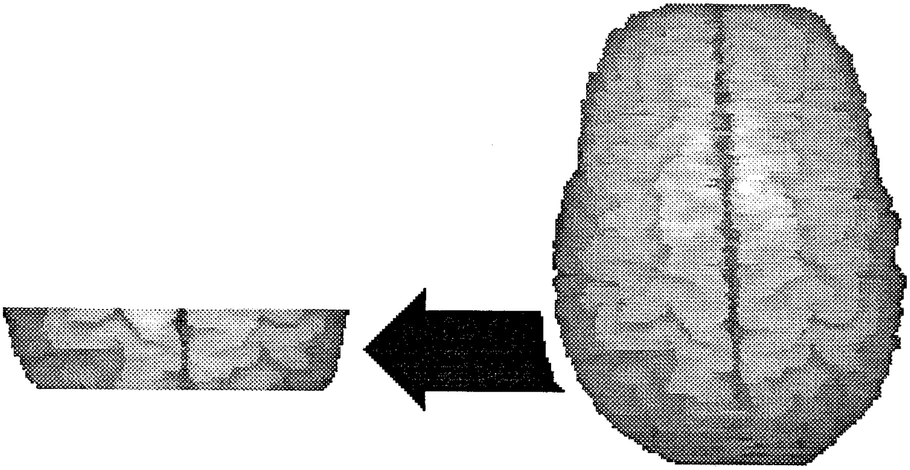

FIG. 1. Reconstruction of a piece of the top view of the posterior third of the brain cortex by means of TIS. The reconstruction, showed in the left side, was performed on the basis of the coordinates of 256 sample points on each one of 30 serial cross-sections that form the piece of the human cortex indicated by the tail of the arrow.

u_i exists and is bounded on U, for $k_i = 0, ..., 2m + 1$ and $i = 1, \cdots, p$.

Let $k_i = 0, 2, ..., 2m$ and $i = 1, ..., p$. By definition $(W^{k_1}_{1,m}\psi)(0, u_2, ..., u_p) = \psi^{k_1}(0, u_2, ..., u_p)$ and $(W^{k_1}_{1,m}\psi)(1, u_2, ..., u_p) = \psi^{k_1}(1, u_2, ..., u_p)$, so that

$$h^{k_1}_{1,m}(0, u_2, ..., u_p) = \psi^{k_1}(0, u_2, ..., u_p) \tag{I.3}$$

and

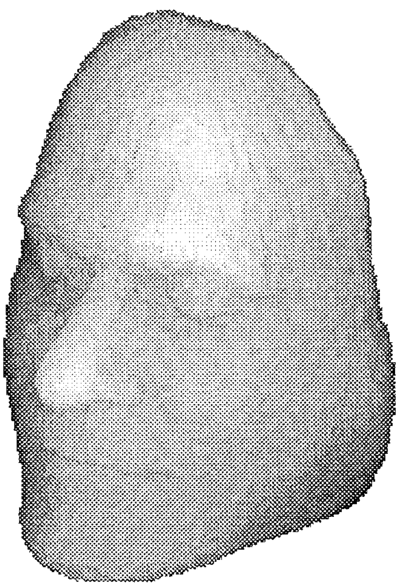

FIG. 2. Reconstruction of the surface of a human face by means of TIS on the basis of the coordinates of 85 sample points on each of 85 serial cross-sections.

$$h^{k_1}_{1,m}(1, u_2, ..., u_p) = \psi^{k_1}(1, u_2, ..., u_p). \tag{I.4}$$

By definition, $(W^{k_2}_{2,m}[\psi - h_{1,m}])(u_1, 0, u_3, ..., u_p) = \psi^{k_2}(u_1, 0, u_3, ..., u_p) - h^{k_2}_{1,m}(u_1, 0, u_3, ..., u_p)$ and $(W^{k_2}_{1,m}[\psi - h_{1,m}])(u_1, 1, u_3, ..., u_p) = \psi^{k_2}(u_1, 1, u_3, ..., u_p) - h^{k_2}_{1,m}(u_1, 1, u_3, ..., u_p)$. Thus,

$$h^{k_2}_{2,m}(u_1, 0, u_3, ..., u_p) = \psi^{k_2}(u_1, 0, u_3, ..., u_p)$$

and

$$h^{k_2}_{2,m}(u_1, 1, u_3, ..., u_p) = \psi^{k_2}(u_1, 1, u_3, ..., u_p).$$

Furthermore, since (I.3) and (I.4) hold, then

$$(W^{k_1}_{2,m}[\psi - h_{1,m}])(0, u_2, ..., u_p) = 0$$

and

$$(W^{k_1}_{2,m}[\psi - h_{1,m}])(1, u_2, ..., u_p) = 0,$$

and so

$$h^{k_1}_{2,m}(0, u_2, ..., u_p) = \psi^{k_1}(0, u_2, ..., u_p)$$

and

$$h^{k_1}_{2,m}(1, u_2, ..., u_p) = \psi^{k_1}(1, u_2, ..., u_p).$$

Likewise, for all $n = 1, ..., p$ and $i \leq n$, it is obtained that

$$h_{n,m}^{k_i}(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_p) = \psi^{k_i}(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_p)$$

and

$$h_{n,m}^{k_i}(u_1, ..., u_{i-1}, 1, u_{i+1}, ..., u_p) = \psi^{k_i}(u_1, ..., u_{i-1}, 1, u_{i+1}, ..., u_p)$$

From this, setting in particular $n = p$, the relations (I.1) and (I.2) are obtained.

Since ϕ is the odd extension on V of a function with $2m + 1$ bounded derivates with respect to the variable v_i ($i = 1, ..., p$), then ϕ has $2m + 1$ bounded derivates with respect to same variable, if

$$\phi^{k_i}(v_1, ..., v_{i-1}, 0-, v_{i+1}, ..., v_p) = \phi^{k_i}(v_1, ..., v_{i-1}, 0+, v_{i+1}, ..., v_p)$$

for all $k_i = 0, ..., 2m + 1$. Here the left and right members denote, respectively, the left and the right k_ith derivatives with respect to the variable v_i at $v_i = 0$. And this last condition follows from the oddness of ϕ and (I.1).

The property (C4) follows from the oddness of ϕ and (I.2).

APPENDIX II: THE UPPER BOUND FOR THE ERROR OF THE APPROXIMATION OF A MULTIDIMENSIONAL FUNCTION BY MEANS OF TIS; THE CASE IN WHICH THE FUNCTION AND ITS DERIVATIVES ARE KNOWN ON THE BOUNDARY

Define the real-valued function $I_i\phi$ on V given by

$$(I_i\phi)(v_1, ..., v_p) = \sum_{k=0}^{2N_i-1} C_k^i(v_1, ..., v_{i-1}, v_{i+1}, ..., v_p) \times \exp(j\pi k(v_i + 1)),$$

where

$$C_k^i(v_1, ..., v_{i-1}, v_{i+1}, ..., v_p) = (1/T_i) \sum_{t=0}^{2N_i-1} \phi(s_1, ..., s_{i-1}, t/N_i - 1, s_{i+1}, ..., s_p) \exp(-j\pi kt/N_i)$$

with $T_i = 2N_i$, $s_i = v_i/N_i - 1$, and $i = 1, ..., p$.

Since, by definition, the function $I_i\phi$ is the approximation by means of TIS to ϕ in the variable v_i, then the function

$$\Phi_i(v_1, ..., v_p) = (I_i(I_{i-1}(... (I_1\phi))))(v_1, ..., v_p)$$

is the i-dimensional TIS that approximates the function ϕ in the variables $v_1, v_2, ..., v_1$. Obviously, $\Phi_p = \Phi$.

By addition and subtraction of the functions Φ_i ($i = 1, ..., p - 1$), it is obtained that

$$|\phi(v) - \Phi(v)| \leq \sum_{i=1}^{p} |\Phi_{i-1}(v) - \Phi_i(v)|,$$

for all v in V, where $\Phi_0 = \phi$. Since Φ_0 satisfies the properties (C3) and (C4) in Section 2.2, then, keeping the variables v_i fixed for $i > 1$, it holds that

$$|\Phi_0(v) - \Phi_1(v)| \leq w_1(\Phi_0, 2\pi/N_1) \log(N_1)/N_1^{2m+1},$$

where $w_1(\Phi_0, 2\pi/N_i)$ is the modulus of continuity of the function Φ_0 with respect to the variable v_1 [7]. Obviously, the right side of this inequality depends on the variables v_i for $i > 1$, but it does not depend on v_1. Furthermore, by definition, for all v_i with $i > 1$ it holds that

$$w_1(\Phi_0, 2\pi/N_1) \leq w_\alpha(\Phi_0, 2\pi/N_1)$$

where the right side is the α-modulus of continuity of the function Φ_0 and the multi-index α is the p-dimensional vector (1, 0, ..., 0) [18]. Since $w_\alpha(\Phi_0, 2\pi/N_1) \leq 2\|\Phi_0\|$ [18], where $\|\Phi_0\| = \max_{v \in V} |\Phi_0(v)|$, then

$$|\Phi_0(v) - \Phi_1(v)| \leq C_1 \log(N_1)/N_1^{2m+1} \tag{II.1}$$

for all v in V, where $C_1 = 2\|\Phi_0\|$ is a constant that does not depend on v.

In the same way it is obtained that

$$|\Phi_1(v) - \Phi_2(v)| \leq 2\|\Phi_1\| \log(N_2)/N_2^{2m+1}.$$

Since $\|\Phi_1\| \leq \|\Phi_0\| + \|\Phi_1 - \Phi_0\|$, Φ_0 is a bounded function, and (II.1) holds for all v in V, then

$$|\Phi_1(v) - \Phi_2(v)| \leq C_2 \log(N_2)/N_2^{2m+1}$$

for all v on V, where C_2 is a constant that does not depend on v.

Likewise, in the same way it is obtained that

$$|\Phi_{i-1}(v) - \Phi_i(v)| \leq C_i \log(N_i)/N_i^{2m+1},$$

for $i = 1, ..., p$ and all v in V. Therefore,

$$|\phi(v) - \Phi(v)| \leq \max_{i=1 \cdots p} \{C_i \log(N_i)/N_i^{2m+1}\}$$

for all v on V.

From this, since $\phi(u) = \psi(u) - h_{p,m}(u)$ and $\Psi(u) = \Phi(u) + h_{p,m}(u)$ for all u on U, it follows $|\phi(u) - \Phi(u)| = |\psi(u) - \Psi(u)|$ for all u on U. So the inequality (2.13) is finally obtained.

APPENDIX III: THE UPPER BOUND FOR THE ERROR OF THE APPROXIMATION OF A MULTIDIMENSIONAL FUNCTION BY MEANS OF TIS; THE CASE IN WHICH THE FUNCTION AND ITS DERIVATIVES ARE KNOWN ONLY AT A SET OF DISCRETE POINTS ON THE BOUNDARY

Let $\tilde{\psi}(u) = \psi(u) - h_{p,m_p}(u) + H_{p,m_p}(u)$ for all u on U. By definition, this function satisfies the following properties:

(i) $\tilde{\psi}(u) = \psi(u)$ for any point u in $K(N_1, ..., N_p)$.

(ii) $\tilde{\psi}^{k_i}(u) = H^{k_i}_{p,m_p}(u)$ for any point u on the boundary of U, $k_i = 0, 2, ..., 2m_p$ and $i = 1, ..., p$.

Thus, the auxiliary function $\tilde{\phi}$, defined on V as the odd prolongation of the function $\tilde{\psi} - H_{p,m_p}$ holds the properties (C3) and (C4) given in Section 2.2. Therefore, if $\tilde{\Psi}(u) = \tilde{\Phi}(u) + H_{p,m_p}(u)$ is the approximation to $\tilde{\psi}(u)$ by means of the TIS $\tilde{\Phi}$ for $\tilde{\phi}$, then $|\tilde{\psi}(u) - \tilde{\Psi}(u)|$ are bounded by the right-hand side member of (2.13); i.e.,

$$|\tilde{\psi}(u) - \tilde{\Psi}(u)| \leq \max_{i=1 \cdots p} \{C_i \log(N_i)/N_i^{2m_p+1}\}$$

for all u in U.

Since condition (i) implies that $\tilde{\Phi}$ is equal to the TIS $\overline{\Phi}$ for the odd extension on V of the function $\psi - H_{p,m_p}$, then $\tilde{\Psi}(u) = \overline{\Phi}(u) + H_{p,m_p}(u)$ for all u on U, and so $\Psi \equiv \tilde{\Psi}$. Moreover, by definition $|\psi(u) - \tilde{\psi}(u)| = |h_{p,m_p}(u) - H_{p,m_p}(u)|$. Therefore,

$$\begin{aligned} |\psi(u) - \Psi(u)| &\leq h_{p,m_p}(u) - H_{p,m_p}(u)| \\ &\quad + |\tilde{\psi}(u) - \tilde{\Psi}(u)| \\ &\leq |h_{p,m_p}(u) - H_{p,m_p}(u)| \\ &\quad + \max_{i=1 \cdots p} \{C_i \log(N_i)/N_i^{2m_p+1}\} \end{aligned}$$

for all u on U.

In turn, this same type of inequality is satisfied by the n-variable ($n = 1, ..., p - 1$) functions of h_{p,m_p} that must be approximated, except, obviously the one-variable functions that satisfies (2.8) with $m = m_1$. Thus, it can be shown (after some rather tedious manipulations) that this recurrent process of approximations leads to (2.15).

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